The Pointlessness of the Expectation Operator in Investing

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J. L. Kelly Jr.'s famous paper *A New Interpretation of the Information Rate (1956)* suggests that the classical statistical expectation operator, $\mathbb{E}[\cdot]$, is fairly useless in an investing context. This paper will study and reinforce that assertion.

1 Kelly's Contributions

No discussion of bet sizing can be had without talking about J. L. Kelly Jr.'s 1956 paper. He wrote this paper while on the clock at Bell Labs. While the paper appears to be a mathematical exploration of information theory, it is really about gambling and investing. He makes some rough tie-ins to information theory for seemingly no other purpose than to convince his employers he was working on something related to telecommunications.

This paper famously coined Kelly Betting, which is often talked about but typically not fully understood by poker players and stock traders from all walks of life. The foundation of Kelly Betting is revolutionary for one reason. Kelly asserted that an investor should not try to maximize the expected value of his capital, but the logarithm of the growth rate of his capital.

Kelly was very fascinated with gambling problems and *beating the house.* He seemed very determined to discover a way to reliably walk out of a casino with more money than he walked in with. We will phrase our discussion of Kelly's work in investment terms rather than gambling terms.

2 The Arithmetic Nature of Expectations

This section serves as a definitional reference for expectation operators and is largely lifted from Wikipedia.

For a discrete random variable X with $i \in 1, ..., n$ possible outcomes, define the possible probabilities as p_i and the outcomes as x_i. The expectation of X is

$$\mathbb{E}[X] = p_1 x_1 + p_2 x_2 + ... + p_n x_n.$$

In other words,

$$\mathbb{E}[X] = \sum_{i=1}^{n} p_i x_i.$$

If the random variable is continuous with probability density function $f(x)$ then the expectation is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

The expectation operator is fundamentally arithmetic rather than geometric. Investing is fundamentally geometric when interest is compounded or gains are reinvested. There are few realistic scenarios where investing is arithmetic. The remainder of this paper will provide confirmatory evidence for this assertion.

3 The Pointlessness of Expectations

Kelly gives a perfect example of the pointlessness of expectation operations in investing with his gambling experiment. Consider a fair-odds game with two outcomes, a win or a loss. The probability of a win is p where $p \in [0, 1]$ and the probability of a loss is $q = 1 - p$. On a win, the investor gets paid $p' = \frac{1}{p}$ times his investment. This satisfies the *fair odds* condition because the payout is the reciprocal of the probability of a win. A fair odds game serves as a good baseline for analytically studying investment behavior, because most real world investment strategies only marginally deviate from fair odds.

Define the account value at time $t \in 0, ..., T$ to be V_t. In this case, the account value is equal to the investor's cash value because the investment is an instantaneous cash-in cash-out transaction.

To illustrate the pointlessness of expectations operations in investing, we will prove the following two assertions.

I. The investor maximizes $\mathbb{E}[V_t]$ for any t by investing his entire account each time. II. The investor loses his entire account with a probability of one as t approaches ∞ by investing his entire account each time.

The primary assertion of this paper is that the expectation of the account value is a poor objective function for optimizing an investment strategy. Proving the previous two assertion independently will prove the primary assertion of this paper.

3.1 Proving Assertion I

First, define V_t as follows, where the investor bets a fraction γ of his capital each time. Let $x_t \in \{0, 1\}$ represent a loss or a win on attempt t.

$$V_1 = (1 - \gamma)V_0 + \gamma V_0 p' x_1$$

Re-arranging terms, we get the following expression which standardizes against the initial capital V_0.

$$V_1 = V_0(1 - \gamma + \gamma p' x_1)$$

Extending time to $t = 2$, we get the following.

$$V_2 = V_0(1 - \gamma + \gamma p' x_1)(1 - \gamma + \gamma p' x_2)$$

Generalizing, we get the following.

$$V_t = V_0 \prod_{i=1}^{t} (1 - \gamma + \gamma p' x_i)$$

If we define the wealth multiplier on a winning investment as $\gamma_+ = 1 + \gamma(p' - 1)$ and define the wealth multiplier on a losing investment as $\gamma_- = 1 - \gamma$, and define the number of wins as $w = \sum_i x_i$, we can re-write the above equation as the following.

$$V_t = V_0 \gamma_+^w \gamma_-^{t-w}$$

Now that we have parameterized V_t in terms of the number of wins, we can write the expectation as follows.

$$\mathbb{E}[V_t] = V_0 \sum_{i=0}^{t} \binom{t}{i} p^{t-i} q^i \gamma_+^{t-i} \gamma_-^i$$

Readers will notice that this resembles the probability mass function of the binomial distribution. Our problem is essentially a weighted binomial expectation.

While it is likely possible, I do not have the faculties or patience to find the following analytically.

$$\underset{\gamma}{\operatorname{argmax}}[\mathbb{E}[V_t]].$$

We will attempt to prove via computer simulation that $\gamma = 1$ maximizes the expectation expression. The following serves as proof via computer simulation.

```python
import numpy as np
from scipy.special import comb as n_choose_k

def compute_the_expectation(
        t: int,
        p: float,
        gamma: float,
        V_0: float,
    ) -> float:

    assert t >= 0
    assert 0 <= p <= 1
    assert 0 <= gamma <= 1
    assert V_0 > 0

    p_prime = 1 / p
    gamma_plus = 1 - gamma + gamma * p_prime
    gamma_minus = 1 - gamma

    accumulator = 0
    for i in range(0, t+1):
        combinatoric = n_choose_k(t, i)
        losses_proba = (1 - p) ** i
        wins_proba = p ** (t - i)
        gamma_minus_part = gamma_minus ** i
        gamma_plus_part = gamma_plus ** (t - i)
        accumulator += (
            combinatoric *
            losses_proba *
            wins_proba *
            gamma_minus_part *
            gamma_plus_part
        )

    return accumulator * V_0
```

```python
# Investment is $1,000,000
V_0 = 1_000_000

# Float tolerance is a one hundredth of a cent
tolerance = 10 ** -4

for t in range(1, 100):
    for p in np.arange(0.01, 1.01, 0.01):
        results = []
        for gamma in np.arange(0.01, 1.01, 0.01):
            result = compute_the_expectation(
                t=t,
                p=p,
                gamma=gamma,
                V_0=V_0,
            )
            results.append(result)

        # The last element is the max
        for result in results[1:]:
            assert results[-1] > (result - tolerance)

        # The max expectation is V_0
        assert (
            (V_0 - tolerance) <=
            results[-1] <=
            (V_0 + tolerance)
        )
```

From this simulation, a curious fact arises that the expectation is always equal to the initial capital when the entire account is invested. I am sure there is an elegant geometric explanation for this, but I will not prove it here. It is also interesting to note that investing in a fair-odds game such as this with anything less than your entire capital reduces your expected capital to something less than what you started with.

3.2 Proving Assertion II

We have succeeded in proving the first condition of our argument that expectation operators are pointless in investing. To close this case, we must also prove that the probability of ruin approaches zero for continuous reinvestment of capital in an expectation maximizing way. This equates to proving the following.

$$\lim_{t\to\infty} \mathbb{P}[\cap_{i=1}^{t}(x_i = 1)] = 0$$

Since

$$\mathbb{P}[\cap_{i=1}^{t}(x_i = 1)] = p^t \ ,$$

we can write

$$\lim_{t\to\infty} p^t = 0 \ ,$$

which concludes our proof for any $0 \leq p < 1$.

4 Conclusion

The expectation is maximized by investing everything, but eventual ruin is also guaranteed by investing everything. Such contradictory properties make the expectation operator wholly inappropriate for financial applications involving compounded or reinvested gains. This fundamental knowledge seems to be largely lost and unappreciated among the investing community. Readers are challenged to think about which common financial performance metrics are rooted in the expectation operator and consider whether or not these metrics are unwise to use as objective functions.